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Research Article

Time-frequency analysis associated with the generalized Gabor transform and applications

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Abstract

In this article, we introduce and study the generalized Gabor transform associated with a class of operators of Sturm-Liouville. We investigate for this transform some problems of time-frequency analysis. In particular, we study the concept of the localization operators and the spectrogram analysis linked to the new generalized Gabor transform. We examine a particular class of the localization operators known as concentration operators. We prove that a finite vector space generated by the first eigenfunctions of such operators has a maximal time-frequency concentration within the region of interest. Then we will use it to approximate functions which are almost concentrated in such a region.

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1. Introduction

Time-Frequency analysis is a powerful technique used across diverse fields of science and engineering to analyze various types of signals and data in both the time and frequency domains simultaneously. This prolific field of harmonic analysis provides a detailed insight into how the frequency components of a signal change over time, making it an essential tool in understanding complex phenomena. The current research proposal aims to explore both the theory and applications of time-frequency analysis in the context of higher-dimensional signals, which are ubiquitous in nature. Typical examples of higher-dimensional signals include the image and video signals: Indeed, a video signal is a sequence of individual video frames, or images. Unlike the one-dimensional signals, there are special problems that arise in higher dimensional signals, such as edge detection, target detection and tracking, and special techniques are required to deal with such problems. As such, it is lucrative to delve into the realm of higher-dimensional time-frequency analysis both from theoretical and application perspectives. The theoretical considerations in the proposal shall encapsulate the formulation of novel tools of time-frequency analysis particularly well suited for higher-dimensional signals arising in numerous scientific pursuits. The application areas shall be primarily encapsulated within the field of signal and image processing with special emphasis in four distinct sectors: Health and wellness, sustainability, energy and industrials, and economics.

The Fourier transform stands out as a significant discovery in mathematical sciences, playing a crucial role in modern scientific and technological advancements. In signal processing, extensive research has utilized the Fourier transform to analyze stationary signals or processes with

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statistically invariant properties over time. However, many signals exhibit non-stationary characteristics, requiring a time-frequency analysis for a comprehensive representation. The Gabor transform, also known as the short-time Fourier transform (STFT), marked a breakthrough in time-frequency analysis. This method involves decomposing non-transient signals using time and frequency-shifted basis functions, termed Gabor window functions. The STFT, with its clear resemblance to the classical Fourier transform, has garnered considerable attention in the past few decades. Its applications span various fields, including harmonic analysis, sampling theory, quantum mechanics, geophysics, astrophysics, medicine, pseudo-differential operators, signal and image processing (cf. [11, 24–26, 31, 35, 36, 38, 45]) and others. Apart from applications, the theoretical skeleton of Gabor transform has likewise been extensively studied and investigated on other groups including hypergroups [16], the locally compact Abelian and non-abelian groups [19–21], Gelfand pairs [43] and so on. For more about Gabor transforms and their applications, we allude to [22, 27, 28].

An hypergroup H is a locally compact topological space with an operation $*$, called convolution, on the space of bounded complex Radon measures on H, which satisfies some axioms. These axioms are chosen to get a good generalization of topological groups. We mention to the Chébli-Trimèche hypergroup, as a fundamental example. The notion of this hypergroup has its origins in the studies of Chébli [13] and Trimèche [41] the harmonic analysis associated with the second-order singular differential operator on $\mathbb{R}_+$, called Sturm-Liouville operator (or Lions operator) [6, 14] and defined by

$$L_A := \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx},$$

where

$$A(x) = x^{2\alpha+1}\,B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

i) For all $x \geq 0, A(x)$ is increasing and $\lim_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\dfrac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho \geq 0$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty), x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x} D(x),$$

where D is a C^∞-function, bounded together with its derivatives.

The substantial development of the Chébli-Trimèche hypergroup had to wait till when Chébli and Trimèche [13, 41] put our hypergroup in the right setting for harmonic analysis. There have been many fruitful developments of the theory of Chébli-Trimèche hypergroup and their applications in various domains of mathematics. Indeed, very recently, many authors have been investigating the behavior of the generalized Fourier transform in the Chébli-Trimèche hypergroup setting, to several problems already studied for the usual Fourier transform; for instance, g-function [2], range of the Chébli-Trimèche transform [3, 8], uncertainty principles [4, 30, 44], maximal functions [7], variation diminishing convolution kernels [10], spectral theorems [13], Paley-Wienr theorems [12, 41], convolution product [13], heat equation [23], Delsarte formula [40], transmutation operators [41], wavelet transform [42], and so on.

As the harmonic analysis in the Chébli-Trimèche hypergroup setting has known remarkable development, the natural question is to ask whether there exists the Gabor transform in this framework. Keeping in view that the localization operators play a vital role in both quantum mechanics and harmonic analysis, our main objective is to present a comprehensive study of localization operators associated with the generalized Gabor transforms. The theory of localization operators has been initiated by Daubechies in [17], developed in series of papers by Wong [9, 46–48], and detailed in the book [49] also by Wong. Nowadays, the localization operators have found many applications to time-frequency analysis, spectral theory and quantum theory. A detailed exposition of different quantization theories and connection between localization operators and Toeplitz operators can be found in [5], (see also [15, 18, 28, 49]).

Motivated by Wong's approaches, the aim of the second part of this paper is to study the boundedness and compactness of localization operators associated with GGT. Our third endeavour is to explore some subjects of the time-frequency analysis associated with the GGT, viz, the spectral analysis for the concentration operators and the spectrogram. It is worth mentioning that the spectrogram plays a vital role in the applications of the Gabor transform to different aspects of signal processing.

The core objectives of this study are described below:

- To introduce and investigate upon fundamental properties of the GGT.
- To study the boundedness and compactness of two-wavelet localization operators associated with the GGT.
- To study the eigenvalues and eigenfunctions of the time-frequency concentration operator.
- To obtain some results on the spectrogram associated with the GGT.

The remainder of this paper is arranged as follows.
In §2, we recall some harmonic analysis results related to the generalized LC Fourier transform studied in [34]. §3 deals to introduce and investigate the generalized Gabor transform. More precisely we prove some properties as the Plancherel's and Lieb's formulas for this transform. In §4, we introduce and study the theory of localization operators associated with GGT. The last section is devoted to investigate the eigenvalues and eigenfunctions of the time-frequency localization operator. Besides, we also study the spectrogram associated with the GGT.

2. Preliminaries

This section gives an introduction to the harmonic analysis associated with the Lions operator. Main references are [13, 41]. To facilitate the narrative, we set some notations as under:

- For $p \in [1, \infty]$, the conjugate exponent shall be denoted by p'.
- $L^p_{\gamma_A}(\mathbb{R}_+)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\begin{aligned} \|f\|_{L^p_{\gamma_A}(\mathbb{R}_+)} &:= \left(\int_{\mathbb{R}_+} |f(x)|^p d\gamma_A(x)\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty \\ \|f\|_{L^\infty_{\gamma_A}(\mathbb{R}_+)} &:= ess \sup_{x \in \mathbb{R}_+} |f(x)| < \infty, \end{aligned}$$

  where

$$d\gamma_A(x) := A(x)dx. \tag{2.1}$$

  For $p = 2$, we provide this space with the scalar product

$$\langle f, g \rangle_{L^2_{\gamma_A}(\mathbb{R}_+)} := \int_{\mathbb{R}_+} f(x)\overline{g(x)} d\gamma_A(x).$$

- $L^p_{\nu_A}(\mathbb{R}_+)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\begin{aligned} \|f\|_{L^p_{\nu_A}(\mathbb{R}_+)} &:= \left(\int_{\mathbb{R}_+} |f(x)|^p d\nu_A(x)\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty \\ \|f\|_{L^\infty_{\nu_A}(\mathbb{R}_+)} &:= ess \sup_{x \in \mathbb{R}_+} |f(x)| < \infty, \end{aligned}$$

  where

$$d\nu_A(\lambda) := \frac{d\lambda}{2\pi |c_A(\lambda)|^2}, \tag{2.2}$$

  c_A here is the Harish-Chandra function associated with L_A. (cf. [13]).
- $C_{c,e}(\mathbb{R})$ the space of even continuous functions on $\mathbb{R}$ with compact support.
- $C_b(\mathbb{R}_+)$ the space of bounded continuous functions on $\mathbb{R}_+$.
- $\mathcal{S}_e(\mathbb{R})$ the Schwartz space of rapidly decreasing even functions on $\mathbb{R}$.

- $S^2(\mathbb{R})$, the generalized Schwartz space defined by

$$S^2(\mathbb{R}) := (\cosh x)^{-2\varrho}\mathcal{S}_e(\mathbb{R}).$$

The topology of the space $S^2(\mathbb{R})$ is given by the following seminorms

$$P_{m,n}(f) = \sup_{\substack{x\in\mathbb{R}\\ 0\leq j\leq n}} (\cosh x)^{2\varrho}(1+x^2)^m|\frac{d^j}{dx^j}f(x)| < \infty.$$

We begin this section, by recalling some properties of the Lions operator L_A.

Proposition 2.1. i) *For each $\lambda\in\mathbb{C}$ the differential equation*

$$L_A u = -(\lambda^2+\varrho^2)u, \quad u(0)=1, \tag{2.3}$$

admits a unique C^∞ solution on $\mathbb{R}_+$, denoted ϕ_λ.
ii) *For every $x\in\mathbb{R}_+$, the function $\lambda\mapsto\phi_\lambda(x)$ is analytic.*
iii) *For all $x,\lambda\in\mathbb{R}_+$, we have*

$$|\phi_\lambda(x)|\leq 1. \tag{2.4}$$

iv) *For all $n\in\mathbb{N}$, $x\in\mathbb{R}_+$ and $\lambda\in\mathbb{C}$, we have*

$$|\frac{d^n}{d\lambda^n}\phi_\lambda(x)|\leq |x|^n e^{|Im\lambda||x|}. \tag{2.5}$$

Remark 2.1. (i) If $A(x)=x^{2\alpha+1}, \alpha>\frac{-1}{2}$, then the differential operator L_A reduced to the so-called Bessel operator. The eigenfunction ϕ_λ is given by

$$\phi_\lambda(x)=j_\alpha(\lambda x),$$

where j_α is the normalized Bessel function defined by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(z/2)^{2k}.$$

.
(ii) If $A(x)=(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha\geq\beta\geq\frac{-1}{2}, \alpha\neq\frac{-1}{2}$, then the differential operator L_A reduced to the so-called Jacobi operator. The eigenfunction ϕ_λ is the Jacobi function of index (α,β) given by

$$\phi_\lambda(x)=F(\frac{1}{2}(\alpha+\beta+1+\lambda),\frac{1}{2}(\alpha+\beta+1-\lambda);\ \alpha+1;\ -(\sinh(x))^2), \tag{2.6}$$

where F is the hypergeometric function ${}_2F_1$ of Gauss.

For $f\in C_{c,e}(\mathbb{R})$, the generalized Fourier transform associated with the Lions operator is defined by

$$\mathcal{F}_A(f)(\lambda)=\int_{\mathbb{R}_+} f(x)\phi_\lambda(x)d\gamma_A(x), \quad \text{for all } \lambda\in\mathbb{C}. \tag{2.7}$$

The inverse Fourier transform of a suitable function g on $\mathbb{R}_+$ is given by:

$$\mathcal{F}_A^{-1}(g)(x)=\int_{\mathbb{R}_+} g(\lambda)\phi_\lambda(x)d\nu_A(\lambda). \tag{2.8}$$

Remark 2.2. i) The function $\lambda\mapsto\frac{1}{|c_A(\lambda)|^2}$ is continuous on $\mathbb{R}_+$.
ii) There exist positive constants R, C_1, C_2 such that

$$C_1|\lambda|^{2\alpha+2}\leq|c_A(\lambda)|^{-2}\leq C_2|\lambda|^{2\alpha+2},$$

when $|\lambda|\geq R$.

Definition 2.3. Let f be in $C_b(\mathbb{R}_+)$ and let $x\in\mathbb{R}_+$. We define the generalized translation operator τ_x^A by

$$\tau_x^A f(y)=\int_{\mathbb{R}_+} f(z)d\mu_{x,y}^A(z) \tag{2.9}$$

where $d\mu_{x,y}^A$ is a measure on $\mathbb{R}_+$ with support in $\Big[|x-y|,x+y\Big]$ (cf. [13,41]).

Now, let us go back to the properties of the generalized translation operator.

Theorem 2.1 ([13,41]). (1) *For all* $f \in L^1_{\mathrm{loc}}(d\gamma_A)$ *and for all* $x, y \in \mathbb{R}_+$, *we have*

$$\tau_x^A f(y) = \tau_y^A f(x) \quad \text{and} \quad \tau_0^A f = f.$$

(2) *For all* $1 \le p \le \infty$ *and* $f \in L^p_{\gamma_A}(\mathbb{R}_+)$, *we have*

$$||\tau_x^A(f)||_{L^p_{\gamma_A}(\mathbb{R}_+)} \le ||f||_{L^p_{\gamma_A}(\mathbb{R}_+)}. \tag{2.10}$$

(3) *If* $f \in L^1_{\gamma_A}(\mathbb{R}_+)$, *and* $x \in \mathbb{R}_+$, *then*

$$\forall\, \lambda \in \mathbb{R}_+, \quad \mathcal{F}_A(\tau_x^A f)(\lambda) = \phi_\lambda^A(x)\mathcal{F}_A(f)(\lambda). \tag{2.11}$$

(4) *Let* $f \in L^1_{\gamma_A}(\mathbb{R}_+)$ *be nonnegative. Then we have*

$$\forall\, x \in \mathbb{R}_+, \quad \tau_x^A f \ge 0.$$

(5) *For any* $f \in L^1_{\gamma_A}(\mathbb{R}_+)$, *we have*

$$\int_{\mathbb{R}_+} \tau_x^A(f)(y) d\gamma_A(y) = \int_{\mathbb{R}_+} f(y) d\gamma_A(y). \tag{2.12}$$

Throughout this paper, $m \in \mathrm{SL}_2(\mathbb{R}_+)$ denotes the matrix parameters such that

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and m^{-1} is its inverse with $b \neq 0$.
Moreover $L^p_{\nu_{A,b}}(\mathbb{R}_+)$ denotes the space of measurable functions, with

$$\|u\|_{L^p_{\nu_{A,b}}(\mathbb{R}_+)} := \left(\int_{\mathbb{R}_+} |u|^p d\nu_{A,b}\right)^{1/p} < \infty, \qquad p \in [1, \infty), \tag{2.13}$$

$$\|u\|_{L^\infty_{\nu_{A,b}}(\mathbb{R}_+)} := \operatorname*{ess\,sup}_{\xi \in \mathbb{R}_+} |u(\xi)| < \infty, \tag{2.14}$$

where

$$d\nu_{A,b}(\xi) := d\nu_A(\frac{\xi}{b}). \tag{2.15}$$

Definition 2.4 ([34]). The generalized canonical Fourier transform $\mathfrak{F}_A^m$ of a function $f \in L^1_{\gamma_A}(\mathbb{R}_+)$ is defined by

$$\mathfrak{F}_A^m(f)(\lambda) = \int_{\mathbb{R}_+} f(x)\, K_A^m(x, \lambda)\, d\gamma_A(x), \quad \text{for all } \lambda \in \mathbb{R}_+, \tag{2.16}$$

where

$$K_A^m(x, \lambda) = \mathrm{e}^{\frac{i}{2}\left(\frac{d}{b}x^2 + \frac{a}{b}\lambda^2\right)} \phi_{\frac{\lambda}{b}}(x). \tag{2.17}$$

We note that we have, for each $x, \lambda \in \mathbb{R}_+$,

$$|K_A^m(x, \lambda)| \le 1. \tag{2.18}$$

Particular cases

• In the case $m := m(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$, $\tau \in \mathbb{R}_+$, $\mathfrak{F}_A^m$ coincides with the Fresnel transform associated with the generalized Fourier transform:

$$\mathcal{W}_A^\tau f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}_+} E_A^\tau(x, y) f(y) d\gamma_A(y), & \tau \neq 0, \\ f(x), & \tau = 0, \end{cases}$$

where $E_A^\tau(x, y) = e^{\frac{i}{2\tau}(x^2+y^2)} \phi_{\frac{y}{\tau}}(x)$. (See [34])

• In the case $m := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathfrak{F}_A^m$ coincides with the generalized Fourier transform $\mathcal{F}_A$, (see [34]).

• When $m := m(\theta) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}$, $\theta \in \mathbb{R}_+$, $\mathfrak{F}_A^m$ coincides with the following integral transform:

$$\mathcal{V}_A^\theta f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}_+} R_A^\theta(x,y) f(y) d\gamma_A(y), & \theta \neq 0, \\ f(x), & \theta = 0, \end{cases}$$

where $R_A^\theta(x,y) = e^{\frac{i}{2}\coth(\theta)(x^2+y^2)} \phi_{\frac{y}{\sinh(\theta)}}(x)$. (See [34])

• In the case $m := m(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, $\theta \in \mathbb{R}_+$, $\mathfrak{F}_A^m$ coincides with the generalized fractional Fourier transform $\mathcal{F}_A^\theta$ (see [34]):

$$\mathcal{F}_A^\theta f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}_+} \mathcal{K}_A^\theta(x,y) f(y) d\gamma_A(y), & (2j-1)\pi < \theta < (2j+1)\pi, \\ f(x), & \theta = 2j\pi, \\ f(-x), & \theta = (2j+1)\pi, \end{cases}$$

where

$$\mathcal{K}_A^\theta(x,y) = e^{-\frac{i}{2}cot(\theta)(x^2+y^2)} \phi_{\frac{y}{\sin(\theta)}}(x).$$

Proposition 2.2 ([34]). *Let $m \in \mathrm{SL}_2(\mathbb{R})$. The differential operator Δ_A^m of Sturm-Liouville type defined by*

$$\Delta_A^m f = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f + \left(\frac{A'(x)}{A(x)} - 2i\frac{d}{b}x \right) \frac{\mathrm{d}}{\mathrm{d}x} f - \frac{d}{b} \left(i + \frac{d}{b}x^2 + ix\frac{A'(x)}{A(x)} \right) f, \tag{2.19}$$

have the following properties

(i) *The operator Δ_A^m is related to the Lions operator L_A by*

$$\mathbf{M}_{-\frac{d}{b}} \circ \Delta_A^m \circ \mathbf{M}_{\frac{d}{b}} = L_A, \tag{2.20}$$

where $\mathbf{M}_s$ is defined by $\mathbf{M}_s f(x) = e^{\frac{is}{2}x^2} f(x)$.

(ii) *The kernel $K_A^m(x,\lambda)$ of the generalized canonical Fourier transform $\mathfrak{F}_A^m$ is the unique solution of*

$$\begin{cases} \Delta_A^m K_A^m(\cdot,\lambda) = -\frac{\lambda^2}{b^2} K_A^m(\cdot,\lambda) \\ \\ K_A^m(0,\lambda) = \mathrm{e}^{\frac{i}{2}\frac{a}{b}\lambda^2}, \ \frac{\mathrm{d}}{\mathrm{d}x} K_A^m(0,\lambda) = 0. \end{cases} \tag{2.21}$$

(iii) *For any $f \in \mathcal{S}_2(\mathbb{R}_+)$ and $g \in C_b^2(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}_+} \Delta_A^m f(x) \overline{g(x)}\, d\gamma_A(x) = \int_{\mathbb{R}_+} f(x) \overline{\Delta_A^m g(x)} d\gamma_A(x). \tag{2.22}$$

Theorem 2.2. (Riemann–Lebesgue's lemma). *Let $m \in \mathrm{SL}_2(\mathbb{R})$. For all $f \in L^1_{\gamma_A}(\mathbb{R}_+)$, the generalized canonical Fourier transform $\mathfrak{F}_A^m(f)$ belongs to $C_0(\mathbb{R}_+)$ and satisfies*

$$\|\mathfrak{F}_A^m(f)\|_{L^\infty(\mathbb{R}_+)} \leq \|f\|_{L^1_{\gamma_A}(\mathbb{R}_+)}. \tag{2.23}$$

Theorem 2.3. *(**Inversion formula**). If $f \in L^1_{\gamma_A}(\mathbb{R}_+)$ such that $\mathfrak{F}_A^m(f) \in L^1_{\nu_{A,b}}(\mathbb{R}_+)$. Then we have the following inversion formula for $\mathfrak{F}_A^m$*

$$\forall\, x \in \mathbb{R}_+, \quad f(x) = \int_{\mathbb{R}_+} \mathfrak{F}_A^m(f)(\xi) \overline{K_A^m(x,\xi)}\, d\nu_A\left(\frac{\xi}{b}\right). \tag{2.24}$$

Theorem 2.4. *(**Parseval's formula**). (i) Let $f, g \in \mathcal{S}_2(\mathbb{R})$. Then we have*

$$\int_{\mathbb{R}_+} f(x) \overline{g(x)} d\gamma_A(x) = \int_{\mathbb{R}_+} \mathfrak{F}_A^m(f)(\lambda) \overline{\mathfrak{F}_A^m(g)(\lambda)}\, d\nu_A\left(\frac{\lambda}{b}\right). \tag{2.25}$$

In particular, we derive the Plancherel's formula as follows

$$\int_{\mathbb{R}_+} |f(x)|^2 d\gamma_A(x) = \int_{\mathbb{R}_+} |\mathfrak{F}_A^m(f)(\lambda)|^2\, d\nu_A\left(\frac{\lambda}{b}\right).$$

(ii) The generalized linear canonical Fourier transform $\mathfrak{F}_A^m$ extends to an isometric isomorphism from $L^2_{\gamma_A}(\mathbb{R}_+)$ onto $L^2_{\nu_{A,b}}(\mathbb{R}_+)$. Moreover, for any $f \in L^2_{\gamma_A}(\mathbb{R}_+)$, we have

$$\int_{\mathbb{R}_+} |f(x)|^2 d\gamma_A(x) = \int_{\mathbb{R}_+} |\mathfrak{F}_A^m(f)(\lambda)|^2 \, d\nu_A\left(\frac{\lambda}{b}\right). \tag{2.26}$$

3. Generalized Gabor transform

This section is devoted to introduce and study the GGT.

Notation. We denote by

- $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}_+$.
- $d\mu_{A,b}(x,\lambda) := d\nu_{A,b}(x) d\gamma_A(\lambda)$, for all $(x,\lambda) \in \mathbb{R}^2_+$.
- $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^2_+$ satisfying

$$\begin{aligned} \|f\|_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)} &:= \Big(\int_{\mathbb{R}^2_+} |f(x,\lambda)|^p d\mu_{A,b}(x,\lambda)\Big)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)} &:= \operatorname*{ess\,sup}_{(x,\lambda)\in\mathbb{R}^2_+} |f(x,\lambda)| < \infty, \quad p = \infty\cdot \end{aligned}$$

We introduce the generalized convolution product $u \underset{A,m}{*} v$ of u, v in $\mathcal{S}_e(\mathbb{R})$ by

$$\forall\, \xi \in \mathbb{R}_+, \qquad u \underset{A,m}{*} v(\xi) := \int_{\mathbb{R}_+} (\mathfrak{F}_A^m)^{-1}(u)(x)(\mathfrak{F}_A^m)^{-1}(v)(x) K_A^m(x,\xi) d\gamma_A(x) \tag{3.1}$$

$$= \mathfrak{F}_A^m\left((\mathfrak{F}_A^m)^{-1}(u)(\mathfrak{F}_A^m)^{-1}(v)\right)(\xi). \tag{3.2}$$

We proceed as in [43], we get

(1) If $u \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ and $v \in L^1_{\nu_{A,b}}(\mathbb{R}_+)$, then $u \underset{A,m}{*} v$ is in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$, with

$$(\mathfrak{F}_A^m)^{-1}\left(u \underset{A,m}{*} v\right) = (\mathfrak{F}_A^m)^{-1}(u)(\mathfrak{F}_A^m)^{-1}(v). \tag{3.3}$$

(2) If $u, v \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$, then $u \underset{A,m}{*} v$ belongs to $L^2_{\nu_{A,b}}(\mathbb{R}_+)$ if and only if $(\mathfrak{F}_A^m)^{-1}(u)(\mathfrak{F}_A^m)^{-1}(v) \in L^2_{\gamma_A}(\mathbb{R}_+)$ such that

$$(\mathfrak{F}_A^m)^{-1}\left(u \underset{A,m}{*} v\right) = (\mathfrak{F}_A^m)^{-1}(u)(\mathfrak{F}_A^m)^{-1}(v). \tag{3.4}$$

(3) If $u, v \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$, then

$$\int_{\mathbb{R}_+} \left|u \underset{A,m}{*} v(\xi)\right|^2 d\nu_{A,b}(\xi) = \int_{\mathbb{R}_+} \left|(\mathfrak{F}_A^m)^{-1}\left(u \underset{A,m}{*} v\right)(\xi)\right|^2 d\gamma_A(\xi). \tag{3.5}$$

The modulation of $\varphi \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ by the real number s is given by

$$\varphi_s^{A,m} := \mathfrak{F}_A^m\left(\sqrt{\tau_s^A\left(\left|(\mathfrak{F}_A^m)^{-1}(\varphi)\right|^2\right)}\right). \tag{3.6}$$

Then

$$\|\varphi_s^{A,m}\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \|\varphi\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{3.7}$$

For $\varphi \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ and $u \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$, we define the generalized Gabor transform (GGT), by

$$\mathfrak{G}_\varphi^{A,m}(u)(t,s) := u \underset{A,m}{*} \varphi_s^{A,m}(t) = \langle u, \varphi_{t,s}^{A,m}\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}, \tag{3.8}$$

where $\varphi_{t,s}^{A,m}$ is defined by

$$(\mathfrak{F}_A^m)^{-1}(\varphi_{t,s}^{A,m})(\xi) := (\mathfrak{F}_A^m)^{-1}(\varphi_s^{A,m})(\xi) K_A^m(s,\xi). \tag{3.9}$$

Proposition 3.1. *For u and $\varphi \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$, we have*

$$\|\mathfrak{G}_\varphi^{A,m}(u)\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)} \le \|u\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \|\varphi\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{3.10}$$

Proof. The result is immediately from relation (3.8), (3.1), Cauchy-Schwarz inequality, (2.4), Plancherel's formula (2.26) and (3.7). □

Proposition 3.2. *(Plancherel's formula) Let φ be in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$. Then, for all u in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$*

$$||\mathfrak{G}^{A,m}_{\varphi}(u)||_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} = \|\varphi\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\|u\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{3.11}$$

Proof. Using relation (3.8), Plancherel's formula (2.26), relation (3.4), Fubini's theorem, (3.6), and (2.12), we get

$$\begin{aligned}
\int_{\mathbb{R}^2_+} |u *_{A,m} \varphi^{A,m}_s(y)|^2 d\nu_{A,b}(y)d\gamma_A(s) &= \int_{\mathbb{R}^2_+} |(\mathfrak{F}^m_A)^{-1}(u)(\xi)|^2 |(\mathfrak{F}^m_A)^{-1}(\varphi^{A,m}_s)(\xi)|^2 d\gamma_A(\xi)d\gamma_A(s)\\
&= \int_{\mathbb{R}^2_+} |(\mathfrak{F}^m_A)^{-1}(u)(\xi)|^2 \tau^A_s(|(\mathfrak{F}^m_A)^{-1}(\varphi)|^2)(\xi)d\gamma_A(\xi)d\gamma_A(s)\\
&= \int_{\mathbb{R}^2_+} |(\mathfrak{F}^m_A)^{-1}(u)(\xi)|^2 \tau^A_\xi(|(\mathfrak{F}^m_A)^{-1}(\varphi)|^2)(s)d\gamma_A(s)d\gamma_A(\xi)\\
&= ||u||^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}||\varphi||^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned}$$

□

As in the classical case, the GGT preserves the orthogonality relation. However, we have the following result.

Corollary 3.1. *Let φ be in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$. Then, for all u, v in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}^2_+} \mathfrak{G}^{A,m}_{\varphi}(u)(y,s)\overline{\mathfrak{G}^{A,m}_{\varphi}(v)(y,s)}d\mu_{A,b}(y,s) = ||\varphi||^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\int_{\mathbb{R}_+} u(x)\overline{v(x)}d\nu_{A,b}(x). \tag{3.12}$$

Proposition 3.3. *Let φ be in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$. For any u in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$ and for all $p \in [2,\infty)$, we have*

$$\|\mathfrak{G}^{A,m}_{\varphi}(u)\|_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)} \leqslant \|u\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\|\varphi\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{3.13}$$

Proof. Using Proposition 3.1 and Proposition 3.2, the result follows by applying the Riesz-Thorin interpolation theorem. □

By simple calculations we prove the following:

Lemma 3.1. *Let $\varphi \in L^2_{\nu_{A,b}}(\mathbb{R}_+)\bigcap L^1_{\nu_{A,b}}(\mathbb{R}_+)$, then for any $u \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$, we have*

$$(\mathfrak{F}^m_A)^{-1}\Big(\mathfrak{G}^{A,m}_{\varphi}(u)(.,s)\Big)(\xi) = (\mathfrak{F}^m_A)^{-1}(u)(\xi)\sqrt{\tau^A_s(|(\mathfrak{F}^m_A)^{-1}(\varphi)|^2)(\xi)}. \tag{3.14}$$

We close this subsection by giving the following:

Proposition 3.4. *Let φ be in $L^2_{\nu_{A,b}}(\mathbb{R}_+)\cap L^1_{\nu_{A,b}}(\mathbb{R}_+)$. Then, $\mathfrak{G}^{A,m}_{\varphi}(L^2_{\nu_{A,b}}(\mathbb{R}_+))$ is a reproducing kernel Hilbert space in $L^2_{\nu_{A,b}}(\mathbb{R}_+)$ with kernel function*

$$\mathfrak{K}^{A,m}_{\varphi}(x',s';x,s) := \frac{1}{\|\varphi\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}}\int_{\mathbb{R}_+} \varphi^{A,m}_{x',s'}(y)\overline{\varphi^{A,m}_{x,s}(y)}d\nu_{A,b}(y). \tag{3.15}$$

Moreover, the kernel is pointwise bounded:

$$|\mathfrak{K}^{A,m}_{\varphi}(x',s';x,s)| \leq 1; \ \ \forall\,(x',s'),\,(x,s) \in \mathbb{R}^2_+. \tag{3.16}$$

Proof. Let $f \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$. We have

$$\mathfrak{G}^{A,m}_{\varphi}(u)(x,s) = \int_{\mathbb{R}_+} u(y)\overline{\varphi^{A,m}_{x,s}(y)}d\nu_{A,b}(y),\ (x,s) \in \mathbb{R}^2_+.$$

Using Parseval's relation (3.12), we obtain

$$\mathfrak{G}^{A,m}_{\varphi}(u)(x,s) = \frac{1}{\|\varphi\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}}\int_{\mathbb{R}^2_+} \mathfrak{G}^{A,m}_{\varphi}(u)(x',s')\overline{\mathfrak{G}^{A,m}_{\varphi}(\varphi^{A,m}_{x,s})(x',s')}d\mu_{A,b}(x',s').$$

On the other hand, using (3.4) one can easily see that for every $(x,s),(x',s') \in \mathbb{R}_+^2$, the function

$$x' \mapsto \frac{1}{\|\varphi\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}} \mathfrak{G}^{A,m}_{\varphi}(\varphi^{A,m}_{x,s})(x',s') = \frac{1}{\|\varphi\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}} \int_{\mathbb{R}_+} \varphi^{A,m}_{x',s'}(y)\overline{\varphi^{A,m}_{x,s}(y)} d\nu_{A,b}(y)$$

belongs to $L^2_{\nu_{A,b}}(\mathbb{R}_+)$. Therefore, by using (3.9), (2.26) and (2.4) we obtain (3.16). □

4. Localization operators associated to the generalized Gabor transform

We denote by $B(L^2_{\nu_A,b}(\mathbb{R}_+))$ the space of bounded operators from $L^2_{\nu_A,b}(\mathbb{R}_+)$ into itself.

Definition 4.1. (i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{\nu_A,b}(\mathbb{R}_+))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.
(ii) The Schatten class S_1 is the space of all compact operators whose singular values lie in $l^1(\mathbb{N})$. The space S_1 is equipped with the norm

$$||A||_{S_1} := \sum_{n=1}^{\infty} s_n(A). \tag{4.1}$$

Remark 4.2. We note that S_2 is the space of Hilbert-Schmidt operators, whereas S_1 is the space of trace class operators.

Definition 4.3. The trace of an operator A in S_1 is defined by

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \tag{4.2}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_{\nu_A,b}(\mathbb{R}_+)$.

Remark 4.4. If A is positive, then

$$tr(A) = ||A||_{S_1}. \tag{4.3}$$

Moreover, a compact operator A on the Hilbert space $L^2_{\nu_A,b}(\mathbb{R}_+)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} \tag{4.4}$$

for any orthonormal basis $(v_n)_n$ of $L^2_{\nu_A,b}(\mathbb{R}_+)$.

Definition 4.5. We define $S_\infty := B(L^2_{\nu_A,b}(\mathbb{R}_+))$, equipped with the norm,

$$||A||_{S_\infty} := \sup_{v\in L^2_{\nu_A,b}(\mathbb{R}_+):||v||_{L^2_{\nu_A,b}(\mathbb{R}_+)}=1} ||Av||_{L^2_{\nu_A,b}(\mathbb{R}_+)}. \tag{4.5}$$

Definition 4.6. Let u, v be measurable functions on $\mathbb{R}_+$, σ be measurable function on $\mathbb{R}_+^2$, we define the two-window localization operator, associated to the generalized Gabor transform on $\mathbb{R}_+$, noted by $\mathfrak{L}^{A,m}_{u,v}(\sigma)$, on $L^2_{\nu_A,b}(\mathbb{R}_+)$, by

$$\mathfrak{L}^{A,m}_{u,v}(\sigma)(f)(y) = \int_{\mathbb{R}_+^2} \sigma(x,t)\mathfrak{G}^{A,m}_u(f)(x,t)\ v^{A,m}_{x,t}(y) d\mu_{A,b}(x,t),\ y \in \mathbb{R}_+. \tag{4.6}$$

It is often more convenient to interpret the definition of $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^2_{\nu_A,b}(\mathbb{R}_+)$ and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$,

$$\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(f), g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} = \int_{\mathbb{R}_+^2} \sigma(y,s)\mathfrak{G}^{A,m}_u(f)(y,s)\overline{\mathfrak{G}^{A,m}_v(g)(y,s)} d\mu_{A,b}(y,s). \tag{4.7}$$

Proposition 4.1. *The adjoint of linear operator*

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

is $\mathfrak{L}^{A,m}_{v,u}(\overline{\sigma}) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$.

Proof. For all f in $L^2_{\nu_A,b}(\mathbb{R}_+)$ and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$ it immediately follows from (4.7)

$$\begin{aligned}\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(f),g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} &= \int_{\mathbb{R}^2_+}\sigma(y,s)\mathfrak{G}^{A,m}_u(f)(y,s)\overline{\mathfrak{G}^{A,m}_v(g)(y,s)}d\mu_{A,b}(y,s)\\ &= \overline{\int_{\mathbb{R}^2_+}\overline{\sigma(y,s)}\mathfrak{G}^{A,m}_v(g)(y,s)\overline{\mathfrak{G}^{A,m}_u(f)(y,s)}d\mu_{A,b}(y,s)}\\ &= \overline{\langle \mathfrak{L}^{A,m}_{v,u}(\overline{\sigma})(g),f\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}} = \langle f,\mathfrak{L}^{A,m}_{v,u}(\overline{\sigma})(g)\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}.\end{aligned}$$

Thus we get

$$(\mathfrak{L}^{A,m}_{u,v}(\sigma))^* = \mathfrak{L}^{A,m}_{v,u}(\overline{\sigma}). \tag{4.8}$$

□

In what follows, such operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ will be named localization operator for the sake of simplicity.

In the rest of this section we assume that u and v belong to $L^2_{\nu_A,b}(\mathbb{R}_+)$ such that

$$\|u\|_{L^2_{\nu_A,b}(\mathbb{R}_+)} = \|v\|_{L^2_{\nu_A,b}(\mathbb{R}_+)} = 1.$$

4.1. Boundedness for $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ on S_∞

The main result of this subsection is to prove that the linear operators

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

are bounded for all symbols $\sigma \in L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \le p \le \infty$. We first consider this problem for σ in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$ and next in $L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)$ and we then conclude by using interpolation theory.

Proposition 4.2. *Let σ be in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$, then the localization operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}. \tag{4.9}$$

Proof. For every functions f and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$, we have from the relations (4.7) and (3.10),

$$\begin{aligned}|\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(f),g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}| &\leqslant \int_{\mathbb{R}^2_+}|\sigma(y,s)|\,|\mathfrak{G}^{A,m}_u(f)(y,s)|\,|\overline{\mathfrak{G}^{A,m}_v(g)(y,s)}|d\mu_{A,b}(y,s)\\ &\leqslant \|\mathfrak{G}^{A,m}_u(f)\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}\|\mathfrak{G}^{A,m}_v(g)\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}\|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &\leqslant \|f\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|g\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}.\end{aligned}$$

Thus,

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

□

Proposition 4.3. *Let σ be in $L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)$, then the localization operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}. \tag{4.10}$$

Proof. For all functions f and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$, we have from Hölder's inequality

$$\begin{aligned}|\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(f),g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}| &\leqslant \int_{\mathbb{R}^2_+}|\sigma(y,s)|\,|\mathfrak{G}^{A,m}_u(f)(y,s)|\,|\overline{\mathfrak{G}^{A,m}_v(g)(y,s)}|d\mu_{A,b}(y,s)\\ &\leqslant \|\sigma\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}\|\mathfrak{G}^{A,m}_u(f)\|_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\|\mathfrak{G}^{A,m}_v(g)\|_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}.\end{aligned}$$

Using Plancherel's formula for $\mathfrak{G}^{A,m}_u$ and $\mathfrak{G}^{A,m}_v$, given by the relation (3.11), we get

$$|\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(f),g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}| \leqslant \|f\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|g\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\sigma\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Thus,

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

□

We can now associate a localization operator

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

to every symbol σ in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \leq p \leq \infty$ and prove that $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 4.1. *Let σ be in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator*

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+),$$

such that

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Proof. Let f be in $L^2_{\nu_A,b}(\mathbb{R}_+)$. We consider the following operator

$$\mathcal{T} : L^1_{\mu_{A,b}}(\mathbb{R}^2_+ \bigcap L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+) \quad \to \quad L^2_{\nu_A,b}(\mathbb{R}_+),$$

given by

$$\mathcal{T}(\sigma) := \mathfrak{L}^{A,m}_{u,v}(\sigma)(f).$$

Then by Proposition 4.2 and Proposition 4.3

$$||\mathcal{T}(\sigma)||_{L^2_{\nu_A,b}(\mathbb{R}_+)} \leqslant ||f||_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} \tag{4.11}$$

and

$$||\mathcal{T}(\sigma)||_{L^2_{\nu_A,b}(\mathbb{R}_+)} \leq ||f||_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\sigma\|_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)}. \tag{4.12}$$

Therefore, by (4.11), (4.12) and the Riesz-Thorin interpolation theorem (see [[39], Theorem 2] and [[49], Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear operator on $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \leq p \leq \infty$ and we have

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma)(f)||_{L^2_{\nu_A,b}(\mathbb{R}_+)} = ||\mathcal{T}(\sigma)||_{L^2_{\nu_A,b}(\mathbb{R}_+)} \leq ||f||_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\sigma\|_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)}. \tag{4.13}$$

Since (4.13) is true for arbitrary functions f in $L^2_{\nu_A,b}(\mathbb{R}_+)$, then we obtain the desired result. □

4.2. Schatten-von Neumann properties for $\mathfrak{L}^{A,m}_{u,v}(\sigma)$

The main result of this subsection is to prove that, the localization operator

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

is in the Schatten class S_p.

Proposition 4.4. *Let σ be in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$, then the localization operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is in S_2 and we have*

$$\|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_2} \leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Proof. Let $\{r_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\nu_A,b}(\mathbb{R}_+)$. Then by (4.7), Fubini's theorem, Parseval's identity and the relations (3.8) and (4.8), we have

$$\begin{aligned}
\sum_{j=1}^{\infty} ||\mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j)||^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} &= \sum_{j=1}^{\infty} \int_{\mathbb{R}^2_+} \sigma(y,s)\langle r_j,\, u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \overline{\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j),\, v^{A,m}_{y,s}\rangle}_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s) \\
&= \int_{\mathbb{R}^2_+} \sigma(y,s) \sum_{j=1}^{\infty} \langle r_j,\, u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \langle (\mathfrak{L}^{A,m}_{u,v}(\sigma))^*(\, v^{A,m}_{y,s}), r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s) \\
&= \int_{\mathbb{R}^2_+} \sigma(y,s) \langle (\mathfrak{L}^{A,m}_{u,v}(\sigma))^*\, v^{A,m}_{y,s}, u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s).
\end{aligned}$$

Thus, from (4.8) and (4.9), we get

$$\sum_{j=1}^{\infty} ||\mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j)||^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} \leq \int_{\mathbb{R}^2_+} |\sigma(y,s)|\, ||(\mathfrak{L}^{A,m}_{u,v}(\sigma))^*||_{S_\infty} d\mu_{A,b}(y,s) \leq \|\sigma\|^2_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} < \infty. \tag{4.14}$$

So, by (4.14) and Proposition 2.8 in the book [49], by Wong

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 4.5. *Let σ be a symbol in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \leqslant p < \infty$. Then the localization operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is compact.*

Proof. Let σ be in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)\bigcap L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)$ such that $\sigma_n \to \sigma$ in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$ as $n \to \infty$. Then by Theorem 4.1

$$||\mathfrak{L}^{A,m}_{u,v}(\sigma_n) - \mathfrak{L}^{A,m}_{u,v}(\sigma)||_{S_\infty} \leq ||\sigma_n - \sigma||_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Thus, we derive that $\mathfrak{L}^{A,m}_{u,v}(\sigma_n)$ converge to $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ in S_∞ when $n \to \infty$. On the other hand, as by Proposition 4.4, $\mathfrak{L}^{A,m}_{u,v}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is compact. □

Theorem 4.2. *Let σ be in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$. Then $\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$ is in S_1 and we have*

$$\|\widetilde{\sigma}\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} \leqslant \|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}, \tag{4.15}$$

where $\widetilde{\sigma}$ is given by

$$\forall\,(y,s) \in \mathbb{R}^2_+, \quad \widetilde{\sigma}(y,s) = \langle \mathfrak{L}^{A,m}_{u,v}(\sigma)\, u^{A,m}_{y,s},\, v^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}.$$

Proof. Since σ is in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$, by Proposition 4.4, $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ is in S_2, then from the canonical form for compact operators given in [49, Theorem 2.2], there exists an orthonormal basis $\{r_j,\ j = 1, 2...\}$ for the orthogonal complement of the kernel of the operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathfrak{L}^{A,m}_{u,v}(\sigma)|$ and $\{\varphi_j,\ j = 1, 2...\}$ an orthonormal set in $L^2_{\nu_A,b}(\mathbb{R}_+)$, such that

$$\mathfrak{L}^{A,m}_{u,v}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \langle f, r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\varphi_j, \tag{4.16}$$

where $s_j,\ j = 1, 2...$ are the positive singular values of $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ corresponding to r_j. Then, we get

$$\|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j), \varphi_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}.$$

Thus, by Fubini's theorem, Cauchy-Schwarz's inequality, Bessel inequality and (3.8), we get

$$\begin{aligned}
\|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty} \langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j), \varphi_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^2_+} \sigma(y,s)\mathfrak{G}^{A,m}_u(r_j)(y,s)\overline{\mathfrak{G}^{A,m}_v(\varphi_j)(y,s)}d\mu_{A,b}(y,s) \\
&\leqslant \int_{\mathbb{R}^2_+} |\sigma(y,s)| \left(\sum_{j=1}^{\infty} |\mathfrak{G}^{A,m}_u(r_j)(y,s)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |\mathfrak{G}^{A,m}_v(\varphi_j)(y,s)|^2\right)^{\frac{1}{2}} d\mu_{A,b}(y,s) \\
&\leqslant \int_{\mathbb{R}^2_+} |\sigma(y,s)|\, \|\, u^{A,m}_{y,s}\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}\|\, v^{A,m}_{y,s}\|_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s) \\
&\leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}.
\end{aligned}$$

Thus

$$\|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

We now prove that $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ satisfies the first member of (4.15). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$, and using formula (4.16), we get

$$\begin{aligned}|\widetilde{\sigma}(y,s)| &= \Big|\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(\ u^{A,m}_{y,s}),\ v^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\Big| \\ &= \Big|\sum_{j=1}^{\infty} s_j \langle\ u^{A,m}_{y,s}, r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\langle \varphi_j,\ v^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\Big| \\ &\leqslant \frac{1}{2}\sum_{j=1}^{\infty} s_j\Big(\Big|\langle\ u^{A,m}_{y,s}, r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\Big|^2 + \Big|\langle\ v^{A,m}_{y,s}, \varphi_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\Big|^2\Big).\end{aligned}$$

Then from Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2_+} |\widetilde{\sigma}(y,s)| d\mu_{A,b}(y,s) \leq \frac{1}{2}\sum_{j=1}^{\infty} s_j\Big(&\int_{\mathbb{R}^2_+} |\langle\ u^{A,m}_{y,s}, r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}|^2 d\mu_{A,b}(y,s) \\ &+ \int_{\mathbb{R}^2_+} |\langle\ v^{A,m}_{y,s}, \varphi_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}|^2 d\mu_{A,b}(y,s)\Big).\end{aligned}$$

Thus using Plancherel's formula for $\mathfrak{G}^{A,m}_u$, $\mathfrak{G}^{A,m}_v$, we get

$$\int_{\mathbb{R}^2_+} |\widetilde{\sigma}(y,s)| d\mu_{A,b}(y,s) \leq \sum_{j=1}^{\infty} s_j = \|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_1}.$$

The proof is completed. □

Corollary 4.1. *For σ in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$, we have the following trace formula*

$$tr(\mathfrak{L}^{A,m}_{u,v}(\sigma)) = \int_{\mathbb{R}^2_+} \sigma(y,s)\langle\ v^{A,m}_{y,s},\ u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s). \tag{4.17}$$

Proof. Let $\{r_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_{\nu_A,b}(\mathbb{R}_+)$. From Theorem 4.2, the localization operator $\mathfrak{L}^{A,m}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (4.2), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned}tr(\mathfrak{L}^{A,m}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty}\langle \mathfrak{L}^{A,m}_{u,v}(\sigma)(r_j), r_j\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \\ &= \sum_{j=1}^{\infty}\int_{\mathbb{R}^2_+} \sigma(y,s)\langle r_j,\ u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\overline{\langle r_j,\ v^{A,m}_{y,s}\rangle}_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s) \\ &= \int_{\mathbb{R}^2_+} \sigma(y,s)\sum_{j=1}^{\infty}\langle r_j,\ u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}\overline{\langle r_j,\ v^{A,m}_{y,s}\rangle}_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s) \\ &= \int_{\mathbb{R}^2_+} \sigma(y,s)\langle v^{A,m}_{y,s}, u^{A,m}_{y,s}\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s),\end{aligned}$$

and the proof is completed. □

In the following we give the main result of this subsection.

Corollary 4.2. *Let σ be in $L^p_{\mu_{A,b}}(\mathbb{R}^2_+)$, $1 \leqslant p \leqslant \infty$. Then, the localization operator*

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \longrightarrow L^2_{\nu_A,b}(\mathbb{R}_+)$$

is in S_p and we have

$$\|\mathfrak{L}^{A,m}_{u,v}(\sigma)\|_{S_p} \leqslant \|\sigma\|_{L^p_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Proof. The result follows from Proposition 4.3, Theorem 4.2 and by interpolation (See [49, Theorem 2.10 and Theorem 2.11]). □

Remark 4.7. If $u = v$ and if σ is a real valued and nonnegative function in $L^1_{\mu_{A,b}}(\mathbb{R}^2_+)$ then

$$\mathfrak{L}^{A,m}_{u,v}(\sigma) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

is a positive operator. So, by (4.3) and Corollary 4.1

$$||\mathfrak{L}_{u,v}^{A,m}(\sigma)||_{S_1} = \int_{\mathbb{R}_+^2} \sigma(y,s)\, ||u_{y,s}^{A,m}||^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(y,s). \tag{4.18}$$

Now we state a result concerning the trace of products of localization operators.

Corollary 4.3. *Let σ_1 and σ_2 be any real-valued and non-negative functions in $L^1_{\mu_{A,b}}(\mathbb{R}_+^2)$. We assume that $u = v$ and u is a function in $L^2_{\nu_A,b}(\mathbb{R}_+)$ such that $||u||_{L^2_{\nu_A,b}(\mathbb{R}_+)} = 1$. Then, the localization operators $\mathfrak{L}_{u,v}^{A,m}(\sigma_1)$, $\mathfrak{L}_{u,v}^{A,m}(\sigma_2)$ satisfies*

$$\Big|\Big|\Big(\mathfrak{L}_{u,v}^{A,m}(\sigma_1)\,\mathfrak{L}_{u,v}^{A,m}(\sigma_2)\Big)^j\Big|\Big|_{S_1} \quad \leq \quad \Big|\Big|\mathfrak{L}_{u,v}^{A,m}(\sigma_1)\Big|\Big|_{S_1}^j \Big|\Big|\mathfrak{L}_{u,v}^{A,m}(\sigma_2)\Big|\Big|_{S_1}^j,$$

for any natural number j.

Proof. By Theorem 1 in the paper [29] by Liu we know that if O_1 and O_2 are in the trace class S_1 and are positive operators, then

$$\forall\, j \in \mathbb{N}, \quad tr(O_1O_2)^j \leq \Big(tr(O_1)\Big)^j \Big(tr(O_2)\Big)^j.$$

So, if we take $O_1 = \mathfrak{L}_{u,v}^{A,m}(\sigma_1)$, $O_2 = \mathfrak{L}_{u,v}^{A,m}(\sigma_2)$ and we invoke the previous remark, the desired result is obtained and the proof is completed. □

5. Spectral analysis for the generalized concentration operator

5.1. Introduction

In this section, h will be in $L^2_{\nu_A,b}(\mathbb{R}_+)$ such that $||h||_{L^2_{\nu_A,b}(\mathbb{R}_+)} = 1$ and U be a subset of $\mathbb{R}_+^2$ with finite measure $0 < \mu_{A,b}(U) < \infty$.

Definition 5.1. The generalized concentration operator is formally defined on $L^2_{\nu_A,b}(\mathbb{R}_+)$, by

$$\mathcal{L}_h^{A,m}(U)(f)(y) = \int_U \mathfrak{G}_h^{A,m}(f)(x,t)\, h_{x,t}^{A,m}(y) d\mu_{A,b}(x,t),\ y \in \mathbb{R}_+. \tag{5.1}$$

It is easy to see that $\mathcal{L}_h^{A,m}(U)(f) = \mathfrak{L}_{u,v}^{A,m}(\sigma)(f)$, when $u = v = h$ and $\sigma = \mathbb{1}_U$.

It is often more convenient to interpret the definition of $\mathcal{L}_h^{A,m}(U)$ in a weak sense, that is, for f in $L^2_{\nu_A,b}(\mathbb{R}_+)$ and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$,

$$\langle \mathcal{L}_h^{A,m}(U)(f), g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} = \int_U \mathfrak{G}_h^{A,m}(f)(x,t)\overline{\mathfrak{G}_h^{A,m}(g)(x,t)} d\mu_{A,b}(x,t). \tag{5.2}$$

In what follows, such operator $\mathcal{L}_h^{A,m}(U)$ will be named concentration operator for the sake of simplicity.

Proposition 5.1. *The concentration operator $\mathcal{L}_h^{A,m}(U)$ is in S_∞ and we have*

$$||\mathcal{L}_h^{A,m}(U)||_{S_\infty} \leqslant 1. \tag{5.3}$$

Proof. For all functions f and g in $L^2_{\nu_A,b}(\mathbb{R}_+)$, we have from Hölder's inequality

$$\begin{aligned}|\langle \mathcal{L}_h^{A,m}(U)(f), g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}| &\leqslant \int_{\mathbb{R}_+^2} |\mathfrak{G}_h^{A,m}(f)(x,t)||\overline{\mathfrak{G}_h^{A,m}(g)(x,t)}| d\mu_{A,b}(x,t)\\ &\leqslant \|\mathfrak{G}_h^{A,m}(f)\|_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2}\, \|\mathfrak{G}_h^{A,m}(g)\|_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)}.\end{aligned}$$

Using Plancherel's formula for $\mathfrak{G}_h^{A,m}$ given by the relation (3.11), we get

$$|\langle \mathcal{L}_h^{A,m}(U)(f), g\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}| \leqslant \|f\|_{L^2_{\nu_A,b}(\mathbb{R}_+)} \|g\|_{L^2_{\nu_A,b}(\mathbb{R}_+)}.$$

Thus,

$$||\mathcal{L}_h^{A,m}(U)||_{S_\infty} \leqslant 1.$$

□

5.2. Generalized Gabor Toeplitz operator

Definition 5.2. We define the generalized Gabor spectrogram of f as

$$\mathbf{S}_h^{A,m}(f)(x,t) = |\mathfrak{G}_h^{A,m} f(x,t)|^2,\ (x,t) \in \mathbb{R}_+^2. \tag{5.4}$$

Remark 5.3. From the Plancherel formula of $\mathfrak{G}_h^{A,m}$, we have

$$\int_{\mathbb{R}_+^2} \mathbf{S}_h^{A,m}(f)(x,t) d\mu_{A,b}(x,t) = \|f\|^2_{L^2_{\nu_A,b}(\mathbb{R}_+)}. \tag{5.5}$$

It justifies the interpretation of a spectrogram as a time-frequency energy density. Note that also by (5.2)

$$\left\langle \mathcal{L}_h^{A,m}(U) f, f \right\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} = \int_U \mathbf{S}_h^{A,m}(f)(x,t) d\mu_{A,b}(x,t). \tag{5.6}$$

In order to prove a concentration result of the GGT, we need the following notations:
$\mathfrak{P}_h^{A,m} : L^2_{\mu_{A,b}}(\mathbb{R}_+^2) \to L^2_{\mu_{A,b}}(\mathbb{R}_+^2)$ the orthogonal projection from $L^2_{\mu_{A,b}}(\mathbb{R}_+^2)$ onto $\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$.
$P_U : L^2_{\mu_{A,b}}(\mathbb{R}_+^2) \to L^2_{\mu_{A,b}}(\mathbb{R}_+^2)$ the orthogonal projection from $L^2_{\mu_{A,b}}(\mathbb{R}_+^2)$ onto the subspace of function supported in the subset $U \subset \mathbb{R}_+^2$ with

$$0 < \mu_{A,b}(U) := \int_U d\mu_{A,b}(x,t) < \infty.$$

We put

$$||P_U \mathfrak{P}_h^{A,m}|| := \sup\left\{ ||P_U \mathfrak{P}_h^{A,m} v||_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} :\ v \in L^2_{\mu_{A,b}}(\mathbb{R}_+^2),\ ||v||_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} = 1 \right\}.$$

Definition 5.4. We define the generalized Gabor Toeplitz operator

$$T_{h,U}^{A,m} : \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)) \to \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$$

by

$$T_{h,U}^{A,m} F = \mathfrak{P}_h^{A,m} P_U F. \tag{5.7}$$

Proposition 5.2. *The operator $T_{h,U}^{A,m} : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$ is trace-class and satisfies*

$$0 \le T_{h,U}^{A,m} \le P_U \le I, \tag{5.8}$$

and

$$T_{h,U}^{A,m} = \mathfrak{G}_h^{A,m} \mathcal{L}_h^{A,m}(U) (\mathfrak{G}_h^{A,m})^*, \tag{5.9}$$

where $(\mathfrak{G}_h^{A,m})^* : L^2_{\mu_{A,b}}(\mathbb{R}_+^2) \to L^2_{\nu_A,b}(\mathbb{R}_+)$ the adjoint of $\mathfrak{G}_h^{A,m}$ defined by

$$\langle \mathfrak{G}_h^{A,m}(f), g \rangle_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} = \langle f, (\mathfrak{G}_h^{A,m})^*(g) \rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}, \quad f \in L^2_{\nu_A,b}(\mathbb{R}_+),\ g \in L^2_{\mu_{A,b}}(\mathbb{R}_+^2). \tag{5.10}$$

Proof. For all $F \in \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$,

$$\left\langle T_{h,U}^{A,m} F, F \right\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} = \left\langle \mathfrak{P}_h^{A,m}(P_U F), F \right\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} = \langle P_U F, F \rangle_{L^2_{\mu_{A,b}}(\mathbb{R}_+^2)} = \int_U |F(x,t)|^2 d\mu_{A,b}(x,t). \tag{5.11}$$

Thus we deduce (5.8), and $T_{h,U}^{A,m}$ is bounded and positive.
Now, we want to prove (5.9). Indeed, using $\mathfrak{G}_h^{A,m}$ and $(\mathfrak{G}_h^{A,m})^*$, the concentration operator

$$\mathcal{L}_h^{A,m}(U) : L^2_{\nu_A,b}(\mathbb{R}_+) \to L^2_{\nu_A,b}(\mathbb{R}_+)$$

can be written as

$$\mathcal{L}_h^{A,m}(U)(f) = (\mathfrak{G}_h^{A,m})^*(P_U \mathfrak{G}_h^{A,m} f), \quad f \in L^2_{\nu_A,b}(\mathbb{R}_+).$$

Therefore

$$(\mathfrak{G}_h^{A,m} \mathcal{L}_h^{A,m}(U) (\mathfrak{G}_h^{A,m})^*) F = \mathfrak{P}_h^{A,m} P_U F = T_{h,U}^{A,m} F, \quad F \in \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)). \tag{5.12}$$

Therefore the concentration operator $\mathcal{L}_h^{A,m}(U)$ and the generalized Gabor Toeplitz operator $T_{h,U}^{A,m}$ are related by

$$T_{h,U}^{A,m} = \mathfrak{G}_h^{A,m} \mathcal{L}_h^{A,m}(U) (\mathfrak{G}_h^{A,m})^*.$$

□

Remark 5.5. From the above Proposition we deduce that $T_{h,U}^{A,m}$ and $\mathcal{L}_h^{A,m}(U)$ enjoy the same spectral properties, in particular, we have the following proposition.

Proposition 5.3. *The generalized Gabor Toeplitz operator $T_{h,U}^{A,m}$ is compact and even trace class with*

$$tr(T_{h,U}^{A,m}) = tr(\mathcal{L}_h^{A,m}(U)) = \mathfrak{C}_{A,m}(h,U), \tag{5.13}$$

where

$$\mathfrak{C}_{A,m}(h,U) := \int_U \|h_{x,t}^{A,m}\|^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(x,t). \tag{5.14}$$

Proof. We known that the operator $T_{h,U}^{A,m} : \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)) \to \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$ is bounded and positive. Now, let $\{r_n\}_{n=1}^{\infty}$ be an arbitrary orthonormal basis for $\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$. Thus, if we denote by $k_n = (\mathfrak{G}_h^{A,m})^*(r_n)$, then $\{k_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2_{\nu_A,b}(\mathbb{R}_+)$. Thus by (5.2) and Fubini's theorem

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\langle T_{h,U}^{A,m}(r_n), r_n \right\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} &= \sum_{n=1}^{\infty} \left\langle \mathcal{L}_h^{A,m}(U)(\mathfrak{G}_h^{A,m})^*(r_n), (\mathfrak{G}_h^{A,m})^*(r_n) \right\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} \\
&= \sum_{n=1}^{\infty} \int_U |\mathfrak{G}_h^{A,m}(k_n)(x,t)|^2 d\mu_{A,b}(x,t) \\
&= \int_U \sum_{n=1}^{\infty} |\mathfrak{G}_h^{A,m}(k_n)(x,t)|^2 d\mu_{A,b}(x,t) \\
&= \int_U \sum_{n=1}^{\infty} |\left\langle k_n, h_{x,t}^{A,m} \right\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)}|^2 d\mu_{A,b}(x,t) \\
&= \int_U \|h_{x,t}^{A,m}\|^2_{L^2_{\nu_A,b}(\mathbb{R}_+)} d\mu_{A,b}(x,t) \\
&= \mathfrak{C}_{A,m}(h,U).
\end{aligned}$$

Therefore, by Definition 4.3 and Remark 4.4, the operator $T_{h,U}^{A,m}$ is trace class with

$$\|T_{h,U}^{A,m}\|_{S_1} = tr(T_{h,U}^{A,m}) = \mathfrak{C}_{A,m}(h,U).$$

□

Let $\mathbf{V}_{h,U}^{A,m} : L^2_{\mu_{A,b}}(\mathbb{R}^2_+) \to L^2_{\mu_{A,b}}(\mathbb{R}^2_+)$ the operator defined by $\mathbf{V}_{h,U}^{A,m} = \mathfrak{P}_h^{A,m} P_U \mathfrak{P}_h^{A,m}$. The advantage of $\mathbf{V}_{h,U}^{A,m}$ compared to $T_{h,U}^{A,m}$ is that it is defined on $L^2_{\mu_{A,b}}(\mathbb{R}^2_+)$ and consequently its spectral properties can be easily related to its integral kernel. Since $T_{h,U}^{A,m}$ is positive and trace-class, then using the decomposition

$$L^2_{\mu_{A,b}}(\mathbb{R}^2_+) = \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)) \oplus \left(\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))\right)^{\perp},$$

we deduce that $\mathbf{V}_{h,U}^{A,m}$ is also positive and trace-class with

$$tr(\mathbf{V}_{h,U}^{A,m}) = tr(T_{h,U}^{A,m}) = \mathfrak{C}_{A,m}(h,U). \tag{5.15}$$

In addition, we have the following result.

Proposition 5.4. *The trace of $(T_{h,U}^{A,m})^2$ is given by*

$$tr((T_{h,U}^{A,m})^2) = \int_U \int_U |\mathfrak{K}_h^{A,m}(x,t;x',t')|^2 d\mu_{A,b}(x,t) d\mu_{A,b}(x',t'). \tag{5.16}$$

Proof. As $\mathbf{V}_{h,U}^{A,m}$ is positive, then

$$tr((T_{h,U}^{A,m})^2) = tr((\mathbf{V}_{h,U}^{A,m})^2). \tag{5.17}$$

Moreover, as the space $\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$ is a reproducing kernel Hilbert space with kernel $\mathfrak{K}_h^{A,m}$, we get that for $F \in L^2_{\mu_{A,b}}(\mathbb{R}^2_+)$

$$\mathbf{V}_{h,U}^{A,m}F(x,t) = \int_{\mathbb{R}^2_+} F(x',t') \int_{\mathbb{R}^2_+} \mathbb{1}_U(y,s)\mathfrak{K}_h^{A,m}(x,t;y,s)\mathfrak{K}_h^{A,m}(y,s;x',t')d\mu_{A,b}(y,s)d\mu_{A,b}(x',t'). \tag{5.18}$$

That is, $\mathbf{V}_{h,U}^{A,m}$ has integral kernel

$$\mathbf{N}_{h,U}^{A,m}(x,t;x',t') = \int_{\mathbb{R}^2_+} \mathbb{1}_U(y,s)\mathfrak{K}_h^{A,m}(x,t;y,s)\mathfrak{K}_h^{A,m}(y,s;x',t')d\mu_{A,b}(y,s). \tag{5.19}$$

Therefore

$$\begin{aligned} tr((\mathbf{V}_{h,U}^{A,m})^2) &= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} |\mathbf{N}_{h,U}^{A,m}(x,t;x',t')|^2 d\mu_{A,b}(x,t)d\mu_{A,b}(x',t') \\ &= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} \mathbf{N}_{h,U}^{A,m}(x,t;x',t')\overline{\mathbf{N}_{h,U}^{A,m}(x,t;x',t')}d\mu_{A,b}(x,t)d\mu_{A,b}(x',t') \\ &= \int_U\int_U \mathbf{K}_h^{A,m}(z_1;z_2)d\mu_{A,b}(z_1)d\mu_{A,b}(z_2). \end{aligned}$$

Involving the properties of the kernel of the reproducing kernel Hilbert space

$$\begin{aligned} \mathbf{K}_h^{A,m}(z_1;z_2) &= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} \mathfrak{K}_h^{A,m}(z_2;x,t)\mathfrak{K}_h^{A,m}(x,t;z_1)\mathfrak{K}_h^{A,m}(z_1;x',t')\mathfrak{K}_h^{A,m}(x',t';z_2)d\mu_{A,b}(x,t)d\mu_{A,b}(x',t') \\ &= \mathfrak{K}_h^{A,m}(z_2;z_1)\mathfrak{K}_h^{A,m}(z_1;z_2). \end{aligned}$$

By standard analysis, we get

$$\mathbf{K}_h^{A,m}(z_1;z_2) = |\mathfrak{K}_h^{A,m}(z_1;z_2)|^2. \tag{5.20}$$

This follows us to conclude. □

5.3. Eigenvalues and eigenfunctions

Since the concentration operator $\mathcal{L}_h^{A,m}(U) = (\mathfrak{G}_h^{A,m})^*\mathbb{1}_U\mathfrak{G}_h^{A,m}$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$\mathcal{L}_h^{A,m}(U)(f) = \sum_{n=1}^{\infty} s_n^{A,m}(U)\left\langle f, v_n^{A,m}(U)\right\rangle_{L^2_{\nu_A,b}(\mathbb{R}_+)} v_n^{A,m}(U), \quad f \in L^2_{\nu_A,b}(\mathbb{R}_+), \tag{5.21}$$

where $\{s_n^{A,m}(U)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a nonincreasing manner and $\{v_n^{A,m}(U)\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n^{A,m}(U) \searrow 0$ and by Proposition 5.1, we have for all $n \geq 1$,

$$s_n^{A,m}(U) \leq s_1^{A,m}(U) \leq 1. \tag{5.22}$$

This, together with (5.9), we can deduce that the generalized Gabor Toeplitz operator

$$T_{h,U}^{A,m} : \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)) \to \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$$

can be diagonalized as

$$T_{h,U}^{A,m}\, F = \sum_{n=1}^{\infty} s_n^{A,m}(U)\left\langle F, r_n^{A,m}(U)\right\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} r_n^{A,m}(U), \quad F \in \mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+)), \tag{5.23}$$

where $r_n^{A,m}(U) = \mathfrak{G}_h^{A,m}(v_n^{A,m}(U))$.

Theorem 5.1. *For all $z = (x,s) \in \mathbb{R}^2_+$, we have*

$$\Theta_{A,m}(z) := \int_{\mathbb{R}^2_+} \mathbb{1}_U(t)|\mathfrak{K}_h^{A,m}(t;z)|^2 d\mu_{A,b}(t) = \sum_{n=1}^{\infty} s_n^{A,m}(U)\mathbf{S}_h^{A,m}(v_n^{A,m}(U))(z). \tag{5.24}$$

Proof. From Proposition 3.4, we have for all $z=(x,s)\in\mathbb{R}^2_+$, the function $\mathfrak{K}_h^{A,m}(.;z)$ is in $\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$. Therefore using the properties of the kernel of the reproducing kernel Hilbert space, we get

$$\begin{aligned}\Big\langle T_{h,U}^{A,m}\,\mathfrak{K}_h^{A,m}(.;z),\mathfrak{K}_h^{A,m}(.;z)\Big\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} &= \Big\langle P_U\mathfrak{K}_h^{A,m}(.;z),\mathfrak{K}_h^{A,m}(.;z)\Big\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &= \int_{\mathbb{R}^2_+}\mathbb{1}_U(t)\mathfrak{K}_h^{A,m}(t;z)\overline{\mathfrak{K}_h^{A,m}(t;z)}d\mu_{A,b}(t)\\ &= \int_{\mathbb{R}^2_+}\mathbb{1}_U(t)|\mathfrak{K}_h^{A,m}(t;z)|^2d\mu_{A,b}(t)\\ &= \Theta_{A,m}(z).\end{aligned}$$

Let $\{w_n^{A,m}(U)\}_{n=1}^\infty\subset\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$ be an orthonormal basis of $Ker(T_{h,U}^{A,m})$.
Thus, $\{w_n^{A,m}(U)\}_{n=1}^\infty\cup\{r_n^{A,m}(U)\}_{n=1}^\infty$ is an orthonormal basis of $\mathfrak{G}_h^{A,m}(L^2_{\nu_A,b}(\mathbb{R}_+))$ and therefore the reproducing kernel $\mathfrak{K}_h^{A,m}$ can be written as

$$\mathfrak{K}_h^{A,m}(x,s;x',t')=\overline{\mathfrak{K}_h^{A,m}(x',t';z)}=\sum_{n=1}^\infty r_n^{A,m}(U)(z)\overline{r_n^{A,m}(U)(x',t')}+\sum_{n=1}^\infty w_n^{A,m}(U)(z)\overline{w_n^{A,m}(U)(x',t')}. \tag{5.25}$$

Using this, we compute again

$$\begin{aligned}\Theta_{A,m}(z)&=\Big\langle T_{h,U}^{A,m}\,\mathfrak{K}_h^{A,m}(.;z),\mathfrak{K}_h^{A,m}(.;z)\Big\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &=\left\langle T_{h,U}^{A,m}\sum_{n=1}^\infty\overline{r_n^{A,m}(U)(z)}r_n^{A,m}(U),\sum_{k=1}^\infty\overline{r_k^{A,m}(U)(z)}r_k^{A,m}(U)\right\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &=\sum_{n,k}\overline{r_n^{A,m}(U)(z)}r_k^{A,m}(U)(z)\Big\langle T_{h,U}^{A,m}\,r_n^{A,m}(U),r_k^{A,m}(U)\Big\rangle_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &=\sum_{n=1}^\infty s_n^{A,m}(U)|r_n^{A,m}(U)(z)|^2,\end{aligned}$$

and the conclusion follows. □

Let $\varepsilon\in(0,1)$ and define the quantity

$$\mathcal{N}(\varepsilon,U):=card\Big\{j:s_j^{A,m}(U)\geq 1-\varepsilon\Big\}.$$

We apply the same arguments used in [1] we prove the following.

Proposition 5.5. *Let $\varepsilon\in(0,1)$. We have*

$$|\mathcal{N}(\varepsilon,U)-\mathfrak{C}_{A,m}(h,U)|\leq\max\{\frac{1}{\varepsilon},\frac{1}{1-\varepsilon}\}\Big|\int_U\int_U|\mathfrak{K}_h^{A,m}(x',t';x,t)|^2d\mu_{A,b}(x,t)d\mu_{A,b}(x',t')-\mathfrak{C}_{A,m}(h,U)\Big|.$$

5.4. Spectrogram of a subspace

Given an N-dimensional subspace V of $L^2_{\nu_A,b}(\mathbb{R}_+)$, P_V the orthogonal projection onto V with projection kernel k_V, i.e.

$$P_Vf(.)=\int_{\mathbb{R}_+}k_V(.;t)f(t)d\nu_{A,b}(t). \tag{5.26}$$

Recall that if $\{v_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$k_V(r;t)=\sum_{n=1}^N v_n(r)\overline{v_n(t)}. \tag{5.27}$$

The kernel k_V is independent of the choice of orthonormal basis for V.

Definition 5.6. The spectrogram of the space V with respect h is defined as

$$\mathbf{SPEC}_h^{A,m}V(x,t):=\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}k_V(s;y)\overline{h_{x,t}^{A,m}(s)}h_{x,t}^{A,m}(y)d\nu_{A,b}(s)d\nu_{A,b}(y). \tag{5.28}$$

Then we have the following result.

Lemma 5.1. *The spectrogram* $\mathbf{SPEC}_h^{A,m}V$ *is given by*

$$\mathbf{SPEC}_h^{A,m}V = \sum_{n=1}^{N} \mathbf{S}_h^{A,m}(v_n). \tag{5.29}$$

Proof. We have

$$\begin{aligned}
\mathbf{SPEC}_h^{A,m}V(x,t) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sum_{n=1}^{N} v_n(s)\overline{v_n(y)h_{x,t}^{A,m}(s)}h_{x,t}^{A,m}(y)d\nu_{A,b}(s)d\nu_{A,b}(y)\\
&= \sum_{n=1}^{N} \left\langle v_n, h_{x,t}^{A,m}\right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \overline{\left\langle v_n, h_{x,t}^{A,m}\right\rangle}_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\
&= \sum_{n=1}^{N} \mathfrak{G}_h^{A,m}(v_n)(x,t)\overline{\mathfrak{G}_h^{A,m}(v_n)(x,t)}\\
&= \sum_{n=1}^{N} |\mathfrak{G}_h^{A,m}(v_n)(x,t)|^2.
\end{aligned}$$

This allows us to conclude. □

Definition 5.7. We define the time-frequency concentration of a subspace V in U as:

$$\xi_{U,h}^{A,m}(V) := \frac{1}{N}\int_U \mathbf{SPEC}_h^{A,m}V(x,t)d\mu_{A,b}(x,t). \tag{5.30}$$

Then from Lemma 5.1,

$$\xi_{U,h}^{A,m}(V) := \frac{1}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^{A,m}(v_n)(x,t)d\mu_{A,b}(x,t). \tag{5.31}$$

Theorem 5.2. *The* N*-dimensional signal space* $V_N = span\{v_n^{A,m}(U)\}_{n=1}^N$ *consisting of the first* N *eigenfunctions of* $\mathcal{L}_h^{A,m}(U)$ *corresponding to the* N *largest eigenvalues* $\{s_n^{A,m}(U)\}_{n=1}^N$ *maximize the regional concentration* $\xi_{U,h}^{A,m}(V)$ *and*

$$\xi_{U,h}^{A,m}(V_N) := \frac{1}{N}\sum_{n=1}^{N} s_n^{A,m}(U). \tag{5.32}$$

Proof. We have

$$\xi_{U,h}^{A,m}(V_N) := \frac{1}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^{A,m}(v_n^{A,m}(U))(x,t)d\mu_{A,b}(x,t). \tag{5.33}$$

Moreover, the min-max lemma for self-adjoint operators states that (see e. g. Sec.95 in [37])

$$\begin{aligned}
s_n^{A,m}(U) &= \int_U \mathbf{S}_h^{A,m}(v_n^{A,m}(U))(x,t)d\mu_{A,b}(x,t)\\
&= \max\left\{\left\langle \mathcal{L}_h^{A,m}(U)(f), f\right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} : \|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = 1, f \perp v_1^{A,m}(U), ..., v_{n-1}^{A,m}(U)\right\}.
\end{aligned}$$

So the eigenvalues of $\mathcal{L}_h^{A,m}(U)$ determines the number of orthogonal functions that have a well-concentrated spectrogram in U. Thus,

$$\xi_{U,h}^{A,m}(V_N) = \frac{1}{N}\sum_{n=1}^{N} s_n^{A,m}(U). \tag{5.34}$$

The min-max characterization of the eigenvalues of compact operators implies that

$$\sum_{n=1}^{N}\left\langle \mathcal{L}_h^{A,m}(U)(v_n^{A,m}(U)), v_n^{A,m}(U)\right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \max\left\{\sum_{n=1}^{N}\left\langle \mathcal{L}_h^{A,m}(U)v_n, v_n\right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} : \{v_n\}_{n=1}^N \text{ orthonormal}\right\}. \tag{5.35}$$

Therefore any N-dimensional subset V of $L^2_{\nu_A,b}(\mathbb{R}_+)$ cannot to be better concentrated in U than V_N, i.e

$$\xi^{A,m}_{U,h}(V) \leq \xi^{A,m}_{U,h}(V_N). \tag{5.36}$$

The proof is complete. □

Remark 5.8. The time-frequency concentration of a subspace V_N in U satisfies,

$$s^{A,m}_N(U) \leq \xi^{A,m}_{U,h}(V_N) \leq s^{A,m}_1(U) \leq 1. \tag{5.37}$$

5.5. Accumulated spectrogram

Let $\rho^{A,m}_{(h,U)} := \mathbf{SPEC}^{A,m}_h V_{N_{A,m}(h,U)}$, called the accumulated spectrogram, where we assume that $N_{A,m}(h,U) = [\mathfrak{C}_{A,m}(h,U)]$ is the smallest integer greater than or equal to $\mathfrak{C}_{A,m}(h,U)$ and

$$V_{N_{A,m}(h,U)} = span\{v^{A,m}_n(U)\}^{N_{A,m}(h,U)}_{n=1}.$$

Then

$$\rho^{A,m}_{(h,U)}(x,t) = \sum_{n=1}^{N_{A,m}(h,U)} |\mathfrak{G}^{A,m}_h(v^{A,m}_n(U))(x,t)|^2 = \sum_{n=1}^{N_{A,m}(h,U)} |r^{A,m}_n(U)(x,t)|^2. \tag{5.38}$$

Note that

$$\|\rho^{A,m}_{(h,U)}\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} = N_{A,m}(h,U) = \mathfrak{C}_{A,m}(h,U) + O(1).$$

On the other hand, as

$$\sum_{n=1}^{N_{A,m}(h,U)} s^{A,m}_n(U) \leq tr(\mathcal{L}^{A,m}_h(U)) = \mathfrak{C}_{A,m}(h,U)$$

then we can define the quantity

$$E_{A,m}(h,U) := 1 - \frac{\sum_{n=1}^{N_{A,m}(h,U)} s^{A,m}_n(U)}{\mathfrak{C}_{A,m}(h,U)}. \tag{5.39}$$

which satisfies,

$$0 \leq E_{A,m}(h,U) \leq 1. \tag{5.40}$$

Moreover, we have the following result.

Lemma 5.2. *Let $\varepsilon \in (0,1)$. We have*

$$0 \leq E_{A,m}(h,U) \leq 1 - (1-\varepsilon)\min(1, \frac{\mathcal{N}(\varepsilon,U)}{\mathfrak{C}_{A,m}(h,U)}). \tag{5.41}$$

Proof. Let $\varepsilon \in (0,1)$ and define $l_{A,m}(\varepsilon,U,h) = \min(N_{A,m}(h,U), \mathcal{N}(\varepsilon,U))$. It follows that

$$s^{A,m}_n(U) \geq 1-\varepsilon, \quad 1 \leq n \leq l_{A,m}(\varepsilon,U,h). \tag{5.42}$$

As $N_{A,m}(h,U) \geq l_{A,m}(\varepsilon,U,h)$, we get

$$\sum_{n=1}^{N_{A,m}(h,U)} s^{A,m}_n(U) \geq \sum_{n=1}^{l_{A,m}(\varepsilon,U,h)} s^{A,m}_n(U) \geq (1-\varepsilon) l_{A,m}(\varepsilon,U,h). \tag{5.43}$$

Therefore

$$0 \leq E_{A,m}(h,U) \leq 1 - (1-\varepsilon)\frac{l_{A,m}(\varepsilon,U,h)}{\mathfrak{C}_{A,m}(h,U)}. \tag{5.44}$$

As $N_{A,m}(h,U) \geq \mathfrak{C}_{A,m}(h,U)$, we obtain the desired result. □

Consequently when the eigenvalues $\{s^{A,m}_n(U)\}^{\mathcal{N}(\varepsilon,U)}_{n=0}$ are close to 1, then $E_{A,m}(h,U) \to 0$.
We close this subsection by the following result:

Proposition 5.6. *We have*

$$\frac{1}{\mathfrak{C}_{A,m}(h,U)}\|\rho^{A,m}_{(h,U)} - \Theta_{A,m}\|_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} \leq \frac{1}{\mathfrak{C}_{A,m}(h,U)} + 2E_{A,m}(h,U). \tag{5.45}$$

Proof. From Theorem 5.1, we have, for all $z = (x, s) \in U$

$$\rho^{A,m}_{(h,U)}(z) - \Theta_{A,m}(z) = \sum_{n=1}^{\infty}(t_n - s_n^{A,m}(U))|r_n^{A,m}(U)(z)|^2, \tag{5.46}$$

where $t_n = 1$ if $n \leq N_{A,m}(h, U)$ and 0 otherwise. Now since

$$\| \, |r_n^{A,m}(U)|^2||_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} = 1$$

and

$$\sum_{n=1}^{\infty} s_n^{A,m}(U) = \mathfrak{C}_{A,m}(h, U),$$

we obtain

$$\begin{aligned}
\|\rho^{A,m}_{(h,U)} - \Theta_{A,m}||_{L^1_{\mu_{A,b}}(\mathbb{R}^2_+)} &\leq \sum_{n=1}^{\infty} |s_n^{A,m}(U) - t_n| \\
&= \sum_{n=1}^{\infty} s_n^{A,m}(U) - 2 \sum_{n=1}^{N_{A,m}(h,U)} s_n^{A,m}(U) + N_{A,m}(h, U) \\
&= N_{A,m}(h, U) + \mathfrak{C}_{A,m}(h, U) - 2 \sum_{n=1}^{N_{A,m}(h,U)} s_n^{A,m}(U) \\
&\leq 1 + 2\Big(\mathfrak{C}_{A,m}(h, U) - \sum_{n=1}^{N_{A,m}(h,U)} s_n^{A,m}(U)\Big),
\end{aligned}$$

and the estimate (5.45) follows. □

5.6. Uncertainty Inequalities

A nonzero function $u \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ is said to be ε-concentrated on U if

$$\Big\|P_{U^c}\, \mathfrak{G}_h^{A,m}(u)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} \leq \varepsilon \, \|u\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}, \tag{5.47}$$

where $\varepsilon \in [0, 1)$ and $U^c = \mathbb{R}^2_+ \backslash U$. Notice that, if $\varepsilon = 0$ in Inequality (5.47), then supp $\mathfrak{G}_h^{A,m}(u) = U$, and if $\varepsilon \in (0, 1)$, then $\mathfrak{G}_h^{A,m}(u)$ is almost concentrated on U.

Let $\mathcal{E}_h^{\varepsilon}(U)$ be the subset of $L^2_{\nu_{A,b}}(\mathbb{R}_+)$ containing all functions that satisfy (5.47).

Proposition 5.7. *If $f \in \mathcal{E}_h^{\varepsilon}(U)$, then U satisfies*

$$\mu_{A,b}(U) \geq 1 - \varepsilon. \tag{5.48}$$

Proof. We have by Plancherel-type formula (3.11)

$$\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \Big\|\mathbf{1}_U\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} + \Big\|\mathbf{1}_{U^c}\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}.$$

Then

$$\Big\|\mathbf{1}_U\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} \geq \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}(1 - \varepsilon).$$

On the other hand by (3.10)

$$\Big\|\mathbf{1}_U\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} \leq \mu_{A,b}(U)\Big\|\mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^\infty_{\mu_{A,b}}(\mathbb{R}^2_+)} \leq \mu_{A,b}(U)\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.$$

The proof is complete. □

A function $u \in L^2_{\nu_{A,b}}(\mathbb{R}_+)\backslash\{0\}$ is said to be ε-concentrated with respect to an operator O if

$$\|O(u) - u\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \leq \varepsilon\|u\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{5.49}$$

Proposition 5.8. *A function f belongs to $\mathcal{E}_h^{\varepsilon}(U)$ if and only if*

$$\langle \mathcal{L}_h^{A,m}(U)(f), f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \geq (1-\varepsilon)\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}, \tag{5.50}$$

or equivalently

$$\Big\langle (I-\mathcal{L}_h^{A,m}(U))f, f\Big\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \leq \varepsilon\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{5.51}$$

Proof. By (3.11) and (5.47), we have

$$\begin{aligned}\langle \mathcal{L}_h^{A,m}(U)(f), f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= \Big\|\mathbf{1}_U\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &= \Big\|\mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)} - \Big\|\mathbf{1}_{U^c}\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}\\ &= \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \Big\|\mathbf{1}_{U^c}\, \mathfrak{G}_h^{A,m}(f)\Big\|^2_{L^2_{\mu_{A,b}}(\mathbb{R}^2_+)}.\end{aligned}$$

As desired. □

Then we have the following comparison.

Proposition 5.9. *Let $f \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$.*

1. *If the function f belongs to $\mathcal{E}_h^{\varepsilon}(U)$, then f is 2ε-concentrated with respect to the concentration operator $\mathcal{L}_h^{A,m}(U)$.*
2. *If the function f is ε-concentrated with respect to the concentration operator $\mathcal{L}_h^{A,m}(U)$, then f belongs to $\mathcal{E}_h^{\varepsilon+\sqrt{\varepsilon}}(U)$.*

Proof. Using (4.10) and (5.51), we derive that

$$\begin{aligned}\Big\|\mathcal{L}_h^{A,m}(U)f - f\Big\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= \Big\langle \Big(I-\mathcal{L}_h^{A,m}(U)\Big)^2 f, f\Big\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\ &\leq 2\Big\langle \Big(I-\mathcal{L}_h^{A,m}(U)\Big) f, f\Big\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\ &\leq 2\varepsilon\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&2\langle (I-\mathcal{L}_h^{A,m}(U))f, f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\ =\ &\|\mathcal{L}_h^{A,m}(U)f - f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{L}_h^{A,m}(U)f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\ \leq\ &\|\mathcal{L}_h^{A,m}(U)f - f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \Big(\|\mathcal{L}_h^{A,m}(U)f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \|\mathcal{L}_h^{A,m}(U)f - f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\Big)^2\\ -\ &\|\mathcal{L}_h^{A,m}(U)f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\ =\ &2\|\mathcal{L}_h^{A,m}(U)f - f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + 2\|\mathcal{L}_h^{A,m}(U)f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\|\mathcal{L}_h^{A,m}(U)f - f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.\end{aligned}$$

Thus,

$$\langle (I-\mathcal{L}_h^{A,m}(U))f, f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \leq \|\mathcal{L}_h^{A,m}(U)(f)-f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \|\mathcal{L}_h^{A,m}(U)(f)-f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)},$$

and the proof is complete. □

Now let $U_1,\ U_2 \subset \mathbb{R}^2_+$ be two subsets of finite measure and let $h_1, h_2 \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ be such that

$$\|h_1\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \|h_2\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = 1.$$

Theorem 5.3. *Let $\varepsilon_1, \varepsilon_2 \in (0,1)$, such that $\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2} < 1$. If $f \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ is ε_1-concentrated with respect to the concentration operator $\mathcal{L}_{h_1}^{A,m}(U_1)$ and ε_2-concentrated with respect to the concentration operator $\mathcal{L}_{h_2}^{A,m}(U_2)$, then*

$$\mu_{A,b}(U_1)\mu_{A,b}(U_2) \geq 1 - \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}. \tag{5.52}$$

Proof. Involving (4.10) and (5.49), we have

$$\begin{aligned}
&\left\|f-\mathcal{L}_{h_2}^{A,m}(U_2)\mathcal{L}_{h_1}^{A,m}(U_1)f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\
\leq\ &\left\|f-\mathcal{L}_{h_2}^{A,m}(U_2)f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}+\left\|\mathcal{L}_{h_2}^{A,m}(U_2)f-\mathcal{L}_{h_2}^{A,m}(U_2)\mathcal{L}_{h_1}^{A,m}(U_1)f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\
\leq\ &\left\|\mathcal{L}_{h_2}^{A,m}(U_2)f-f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}+\left\|\mathcal{L}_{h_2}^{A,m}(U_2)\right\|_{S_\infty}\left\|\mathcal{L}_{h_1}^{A,m}(U_1)f-f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\
\leq\ &(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2})\|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\mathcal{L}_{h_2}^{A,m}(U_2)\mathcal{L}_{h_1}^{A,m}(U_1)f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &\geq \|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}-\left\|f-\mathcal{L}_{h_2}^{A,m}(U_2)\mathcal{L}_{h_1}^{A,m}(U_1)f\right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\\
&\geq (1-\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2})\,\|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned}$$

Thus by (4.9),

$$\begin{aligned}
0<1-\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2} &\leq \left\|\mathcal{L}_{h_2}^{A,m}(U_2)\mathcal{L}_{h_1}^{A,m}(U_1)\right\|_{S_\infty}\\
&\leq \left\|\mathcal{L}_{h_2}^{A,m}(U_2)\right\|_{S_\infty}\left\|\mathcal{L}_{h_1}^{A,m}(U_1)\right\|_{S_\infty}\\
&\leq \mu_{A,b}(U_1)\mu_{A,b}(U_2).
\end{aligned}$$

The proof is complete. □

Remark 5.9.

(1) Let $\varepsilon_1,\varepsilon_2\in(0,1)$ such that $\varepsilon_1+\sqrt{\varepsilon_1}<1$ and $\varepsilon_2+\sqrt{\varepsilon_2}<1$. If $f\in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ is ε_1-concentrated with respect to the concentration operator $\mathcal{L}_{h_1}^{A,m}(U_1)$ and ε_2-concentrated with respect to the concentration operator $\mathcal{L}_{h_2}^{A,m}(U_2)$, then by Proposition 5.9 and Proposition 5.7, we have

$$\mu_{A,b}(U_1)\mu_{A,b}(U_2)\geq(1-\varepsilon_1-\sqrt{\varepsilon_1})(1-\varepsilon_2-\sqrt{\varepsilon_2}). \tag{5.53}$$

(2) Let $\varepsilon_1,\varepsilon_2\in(0,1)$, such that $\sqrt{2\varepsilon_1}+\sqrt{2\varepsilon_2}<1$. Then if $f\in\mathcal{E}_{h_1}^{\varepsilon_1}(U_1)\cap\mathcal{E}_{h_2}^{\varepsilon_2}(U_2)$, we have from Theorem 5.3

$$\mu_{A,b}(U_1)\mu_{A,b}(U_2)\geq 1-\sqrt{2\varepsilon_1}-\sqrt{2\varepsilon_2}. \tag{5.54}$$

5.7. Approximation of Functions and Error Estimates

If $f\in\mathcal{V}_n=\operatorname{span}\left\{v_j^{A,m}(U)\right\}_{j=1}^n$, then

$$\begin{aligned}
\langle\mathcal{L}_h^{A,m}(U)f,f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= \textstyle\sum_{j=1}^n s_j^{A,m}(U)\left|\langle f,v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\right|^2\\
&\geq s_n^{A,m}(U)\textstyle\sum_{j=1}^n\left|\langle f,v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}\right|^2\\
&= s_n^{A,m}(U)\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned}$$

This means that $f\in\mathcal{E}_h^{1-s_n^{A,m}(U)}(U)$. Thus

$$\mathcal{V}_n\subset\mathcal{E}_h^{1-s_n^{A,m}(U)}(U). \tag{5.55}$$

In the other hand, we have

$$\mathcal{V}_{\mathcal{N}(\varepsilon,U)}\subset\mathcal{E}_h^{\varepsilon}(U). \tag{5.56}$$

Moreover, since

$$\begin{aligned}
\left\langle\mathcal{L}_h^{A,m}(U)\left(\mathcal{L}_h^{A,m}(U)v_j^{A,m}(U)\right),\mathcal{L}_h^{A,m}(U)v_j^{A,m}(U)\right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= (s_j^{A,m}(U))^3\\
&= s_j^{A,m}(U)\left\|\mathcal{L}_h^{A,m}(U)v_j^{A,m}(U)\right\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}
\end{aligned} \tag{5.57}$$

then by (5.50), the function $\mathcal{L}_h^{A,m}(U)(v_j^{A,m}(U))$ is in $\mathcal{E}_h^\varepsilon(U)$, if and only if $j \leq \mathcal{N}(\varepsilon, U)$. Otherwise neither $v_j^{A,m}(U)$ nor $\mathcal{L}_h^{A,m}(U)(v_j^{A,m}(U))$ are in $\mathcal{E}_h^\varepsilon(U)$.
In contrast, functions in $\mathcal{E}_h^\varepsilon(U)$ are not necessarily in $\mathcal{V}_{\mathcal{N}(\varepsilon,U)}$, the next result characterizes functions that are in $\mathcal{E}_h^\varepsilon(U)$.

Theorem 5.4. *A function f belongs to $\mathcal{E}_h^\varepsilon(U)$ if and only if,*

$$\sum_{j=1}^{\mathcal{N}(\varepsilon,U)} (s_j^{A,m}(U) - 1 + \varepsilon) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 \geq \tag{5.58}$$
$$(1-\varepsilon)\|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \sum_{j=1+\mathcal{N}(\varepsilon,U)}^{\infty} (1 - \varepsilon - s_j^{A,m}(U)) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2.$$

where $\mathcal{P}_{\mathrm{Ker}}(g)$ denotes the orthogonal projection of g onto $\mathrm{Ker}(\mathcal{L}_h^{A,m}(U))$.

Proof. The sequence of eigenfunctions $\{v_j^{A,m}(U)\}$ forms an orthonormal subset for $L^2_{\nu_{A,b}}(\mathbb{R}_+)$ possibly incomplete if $\mathrm{Ker}(\mathcal{L}_h^{A,m}(U)) \neq \{0\}$. Then

$$f = \sum_{j=1}^{\infty} \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} v_j^{A,m}(U) + \mathcal{P}_{\mathrm{Ker}}(f). \tag{5.59}$$

Since $\mathcal{L}_h^{A,m}(U)(\mathcal{P}_{\mathrm{Ker}}(f)) = 0$, then

$$\langle \mathcal{L}_h^{A,m}(U)(f), f \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \sum_{j=1}^{\infty} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2. \tag{5.60}$$

It is easy to see that, f belongs to $\mathcal{E}_h^\varepsilon(U)$ if and only if,

$$\langle \mathcal{L}_h^{A,m}(U)(f), f \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \sum_{j=1}^{\infty} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 \geq (1-\varepsilon)\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{5.61}$$

As

$$\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \langle f, f \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \sum_{j\in\mathbb{N}} \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2,$$

we have $f \in \mathcal{E}_h^\varepsilon(U)$ if and only if,

$$\sum_{j=1}^{\infty} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 \geq (1-\varepsilon) \left(\sum_{j\in\mathbb{N}} \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 + \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right) \tag{5.62}$$

Hence we have the conclusion, since the $s_j^{A,m}(U)$s satisfy

$$1 \geq s_1^{A,m}(U) \geq \cdots \geq s^{A,m}_{\mathcal{N}(\varepsilon,U)}(U) \geq 1 - \varepsilon > s^{A,m}_{1+\mathcal{N}(\varepsilon,U)}(U) \geq \cdots > 0. \tag{5.63}$$

The proof is complete. □

Particularly, $f \in L^2_{\nu_{A,b}}(\mathbb{R}_+)$ belongs to $\mathcal{E}_h^{1-s_n^{A,m}(U)}(U)$ if and only if

$$\begin{aligned} \sum_{j=1}^{n-1} (s_j^{A,m}(U) - s_n^{A,m}(U)) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 &\geq s_n^{A,m}(U) \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \\ &+ \sum_{j=1}^{n+1} \left(s_n^{A,m}(U) - s_j^{A,m}(U) \right) \left| \langle f, v_j^{A,m}(U) \rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2. \end{aligned}$$

Although $f \in \mathcal{E}_h^\varepsilon(U)$ doesn't always lie onto a vector space spanned by eigenfunctions, it can be approximated using a finite number of these eigenfunctions.

Theorem 5.5. *If $f \in \mathcal{E}_h^{\varepsilon}(U)$, we have*

$$\left\| f - \sum_{j=1}^{\mathcal{N}(\tilde{\varepsilon},U)} \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} v_j^{A,m}(U) \right\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \leq \sqrt{\frac{\varepsilon}{\tilde{\varepsilon}}}\, \|f\|_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}, \tag{5.64}$$

where $\tilde{\varepsilon}$ is fixed in $(0,1)$.

Proof. Let $\mathcal{P}$ be the orthogonal projection onto $\mathcal{V}_{\mathcal{N}(\tilde{\varepsilon},U)}$. Then

$$\mathcal{P}f = \sum_{j=1}^{\mathcal{N}(\tilde{\varepsilon},U)} \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} v_j^{A,m}(U)$$

and

$$\|f - \mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \sum_{j=1+\mathcal{N}(\tilde{\varepsilon},U)}^{\infty} \left| \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 = \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.$$

Assuming by contradiction that

$$\|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \sum_{j=1}^{\mathcal{N}(\tilde{\varepsilon},U)} \left| \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 < \left(1 - \frac{\varepsilon}{\tilde{\varepsilon}}\right) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}. \tag{5.65}$$

Then

$$\sum_{1+\mathcal{N}(\tilde{\varepsilon},U)}^{\infty} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 < (1-\tilde{\varepsilon}) \left(\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right).$$

Therefore,

$$\begin{aligned}
\langle \mathcal{L}_h^{A,m}(U) f, f\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= \sum_{j=1}^{\infty} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 \\
&< \|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + (1-\tilde{\varepsilon}) \left(\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right) \\
&< (1-\tilde{\varepsilon}) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \tilde{\varepsilon}(1-\varepsilon/\tilde{\varepsilon}) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - (1-\tilde{\varepsilon}) \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \\
&= (1-\varepsilon) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} - (1-\tilde{\varepsilon}) \|\mathcal{P}_{\mathrm{Ker}}(f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \\
&< (1-\varepsilon) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned}$$

Thus $f \notin \mathcal{E}_h^{\tilde{\varepsilon}}(U)$. As

$$\|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \|\mathcal{P}f + (f - \mathcal{P}f)\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} = \|\mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} + \|f - \mathcal{P}f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}, \tag{5.66}$$

then we obtain the desired result. □

Eq. (5.64) implies that,

$$\begin{aligned}
\forall\, f \in \mathcal{E}_h^{\varepsilon}(U), \qquad \left\langle \mathcal{L}_h^{A,m}(U)\mathcal{P}f, \mathcal{P}f \right\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} &= \sum_{j=1}^{\mathcal{N}(\tilde{\varepsilon},U)} s_j^{A,m}(U) \left| \langle f, v_j^{A,m}(U)\rangle_{L^2_{\nu_{A,b}}(\mathbb{R}_+)} \right|^2 \\
&\geq (1-\tilde{\varepsilon}) \left(1 - \frac{\varepsilon}{\tilde{\varepsilon}}\right) \|f\|^2_{L^2_{\nu_{A,b}}(\mathbb{R}_+)}.
\end{aligned} \tag{5.67}$$

Remark 5.10. Involving the relations (2.16), (3.8), (4.6) and the explicit expression of the particular cases of the generalized LCFT presented in section 2, we derive some typical examples of the localization operators associated with the generalized Gabor transforms. More precisely:
(1). In the case $M := M(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$, $\tau \in \mathbb{R}$, our study in this paper cover the time-frequency analysis associated with the generalized Gabor transform in the generalized Fresnel transform setting not studied in the literature.
(2). In the case $M := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, our study in this paper cover the time-frequency analysis associated with the generalized Gabor transform in the Chébli-Trimèche hypergroup setting.

(3). In the case $M := M(\theta) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}$, or $M := M(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ $\theta \in \mathbb{R}$, our study in this paper cover the time-frequency analysis associated with the generalized Gabor transform in the fractional Sturm-Liouville transforms setting not studied in the literature.

6. Conclusion and perspectives

In the present paper we accomplished three major objectives. Firstly, we investigated upon the notion of the generalized Gabor transform associated with the generalized LCFT and studied its harmonic analysis. Secondly, we introduced and studied the localization operators associated with this new transform. Then we investigate the boundedness and compactness of these operators, according to the symbol and the bounded functions. Thirdly, we introduce and study the scalogram associated with the GGT. We show that the signal space consisting of the first eigenfunctions of the localization operator have the maximum localized scalogram in a region of finite measure in the phase-space plane. Then we show that any essentially concentrated function can be approximated by a linear combination of such eigenfunctions. Motivated by subjects studied in [32, 33] in the future works, we will study the quantitative uncertainty principles, and some applications of the theory of the reproducing kernel Hilbert spaces for the GGT.

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